INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 41, Northern Spring 2020 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. Does the region bounded by the parabolas $y = x^2$ and $y = x^2 - 1$ contain a line segment of length greater than 10^6? Both parabolas are in the same plane. (4 points)

2. Alesha finds positive integers a, b and c and tells Borya that there exist unique positive integers x, y and z such that a, b and c are the least common multiples of x and y, of x and z, and of y and z respectively. Prove that Borya can determine c, if Alesha also tells him the values of a and b. (5 points)

3. Is it possible for a tetrahedron to have two cross-sections, one a square of side length at most 1 and the other a square of side length at least 100? (8 points)

4. Ivan invites $2N$ guests to his birthday party. As they arrive, each of the guests is given a black or white hat to wear. There are N hats of each colour. The guests form one or several dancing circles, with at least two guests in each, and such that the colours of the hats alternate within each circle. Prove that Ivan can form dancing circles in exactly $(2N)!$ different ways. (All hats of the same colour are identical. All the guests are distinct.) (9 points)

5. Let $ABCD$ be a cyclic quadrilateral. Suppose the circles with diameters AB and CD intersect at points X_1 and Y_1, the circles with diameters BC and AD intersect at points X_2 and Y_2 and the circles with diameters AC and BD intersect at points X_3 and Y_3. Prove that the lines X_1Y_1, X_2Y_2 and X_3Y_3 are concurrent. (9 points)

6. There are $2n$ consecutive integers written on a blackboard. In each move, they are divided into pairs and each pair is replaced with their sum and their difference, which may be taken to be positive or negative. Prove that no $2n$ consecutive integers can appear on the board again. (10 points)

7. For which k is it possible to colour black a finite number of 1×1 square cells of an infinite grid such that on each horizontal, vertical and diagonal line of squares of the grid there are either exactly k black squares or none at all? (12 points)